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The localisation of energy in general relativity†

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Abstract. The logic of gravitational field energy localisation for static or quasi-static fields is discussed. A particular form of localisation in the case of spherical symmetry is justified by physical considerations. This form coincides with that general form presented by Møller for the case of the Schwarzschild constant-matter-density fluid but differs when one considers other equations of state.

1. Introduction

Even in the absence of gravitational fields, the localisation of field energy is to a certain extent indefinite (Feynman *et al* 1964). With the inclusion of gravitational field energy in general relativity, the indefiniteness is often compounded by confusion, partly because of the role of the equivalence principle. Since the effect of a gravitational field can be annihilated locally by free-fall (equivalence principle) the role of gravitational field energy is often ignored as a matter of principle, or because of the relative weakness of gravitational binding which prevails in most physical situations, or by some curiously fuzzy combination of both.

Through the years, there have been various attempts to develop a truly invariant general prescription for gravitational field energy localisation. However, the ambiguities inherent in the choice of gravitational energy flux vector in the case of radiative fields would appear to preclude the possibility of success for such a programme. For static (or quasi-static) fields in general, or spherically symmetric fields even with time dependence, there is more to be said about physically meaningful concepts and their interpretation. This is the principal focus of the present work, with primary emphasis on the particularly tractable cases which emerge when one is dealing with spherical symmetry.

2. Spherically symmetric fields

The spherically symmetric gravitational field is described, in Schwarzschild coordinates, by the metric form (Landau and Lifshitz 1975)

$$ds^2 = e^{\nu}\, dt^2 - e^{\lambda}\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{1}$$

In vacuum, the components of the metric tensor at r are simply related to the total

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mass m which produces the field:

$$e^{\nu} = e^{-\lambda} = 1 - \frac{2m}{r}. \tag{2}$$

Within the matter distribution, one of the field equations for λ is

$$8\pi T_0{}^0 = -e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) + \frac{1}{r^2}. \tag{3}$$

To avoid a singularity at the origin, λ must vanish at least like r^2 as r approaches zero and hence the solution of equation (3) is

$$e^{-\lambda} = 1 - \frac{8\pi}{r}\int_0^r T_0{}^0 r'^2 \, dr'. \tag{4}$$

If the body has a radius a, then the matching of the interior and exterior metrics implies that

$$m = 4\pi \int_0^a T_0{}^0 r'^2 \, dr' \tag{5}$$

which is precisely the familiar formula of Newtonian theory. However, the integration in equation (5) is with respect to coordinate volume, $4\pi r^2 \, dr$ rather than proper volume $4\pi r^2 e^{\lambda/2} \, dr$. The discrepancy between the energy of the matter

$$m^* = 4\pi \int_0^a T_0{}^0 r'^2 e^{\lambda/2} \, dr' \tag{6}$$

and the total energy, m, is accounted for by gravitational binding energy.

These considerations led us (Cooperstock and Sarracino 1976, 1977) to an identification of

$$\epsilon_p \equiv T_0{}^0 e^{-\lambda/2} \tag{7}$$

as the proper total energy density in terms of Schwarzschild coordinates, because it is precisely the integration of ϵ_p over proper volume which, through equation (5), yields the total energy m. Moreover, the equation of state

$$\epsilon_p = \text{constant} \tag{8}$$

was proposed as a replacement for the Schwarzschild equation of state

$$T_0{}^0 = \text{constant} \tag{9}$$

to represent the limiting configuration of compact spherically symmetric matter distributions consistent with general relativity. This replacement has a direct bearing on the maximum gravitational red shift from the surface of a star and the upper limit to the mass of non-rotating neutron stars.

One might question the logic of a localisation of energy which includes gravitational energy and, granting the logic, the choice of the form for localisation is not entirely obvious. The common critique of localisation *per se* stems from the principle of equivalence which affords a free-fall observer the luxury of abolishing the effects of gravitation in his local domain. However, the question of the existence or lack of existence of the gravitational field can be answered in an absolute manner through considerations of the Riemann tensor. In spite of the utility of the equivalence principle, particularly in guiding Einstein to Riemannian geometry, it has had the unfortunate influence on many investigators to relegate gravitational energy to some nebulous, ephemeral status and then to discard it entirely with free-fall. For the many physical situations where the effect of gravitation is truly negligible, there is no net detrimental effect in this reasoning. However, in the interesting situations where bodies are reaching the limits of compressibility at radii close to $2m$, the role of gravitation assumes vital proportions. For example, the equations of state, equations (8) and (9), lead to maximum gravitational red shifts from the surface of the body of 2·48 and 2·00 respectively. Gravitational energy exists in an absolute sense. If it is taken into account in the tallying of energy, it can make a considerable difference.

If one moves relative to a body, the perception of its energy content is necessarily altered with respect to the rest energy. One cannot demand complete invariance of the localisation of energy density, within or without the realm of general relativity. Therefore, the argument via the equivalence principle, that free-fall locally removes the effect of a gravitational field is a basis for denying gravitational field energy a local role, is necessarily specious. The logical question is simply this: can one ascribe and justify a physically meaningful localisation of total energy density relative to the rest frame of a body?

Misner and Sharp (1964) and Misner (1965) have, in a sense, justified the energy localisation of equation (7) by dynamical considerations. From equation (7), the total energy in the spherically symmetric matter distribution up to a radius r is

$$m(r) = 4\pi \int_0^r T_0{}^0 r'^2 \, \mathrm{d}r'. \tag{10}$$

They have shown that the rate at which work is done by the external matter on the matter within the radius r is precisely the time rate of change of the energy function given by equation (10).

In fact this localisation can be justified without regard to dynamical considerations. Consider a matter distribution which consists of an interior core of radius r, a vacuum region V_I bounded by a spherical shell of outer radius a and inner radius b followed by the exterior vacuum V_E of infinite extent. As before, the metric in V_E is the Schwarzschild solution given by equations (1), (2) and (5).

Within the region V_I, the outer mass shell which is now an exterior shell, has no influence on the metric. Indeed, in the absence of the interior core, the region V_I becomes Minkowski space, as to be expected from the correspondence with Newtonian theory. The metric in V_I derives entirely from the interior core and hence it is again the Schwarzschild metric of equations (1) and (2) where now equation (4) is to be used for the metric components, appropriate to the contributing mass which now extends only to radius r. Thus it is entirely logical to ascribe a localisation of energy, matter plus gravitational, within r as in equation (10),

$$m(r) = 4\pi \int_0^r T_0{}^0 r'^2 \, \mathrm{d}r'$$

because all physical phenomena at r are governed by the energy of equation (10). Allowing V_I to shrink with b approaching r leads to the same conclusion for the continuum. A test particle moving in V_I, will assume the familiar geodesic trajectories with m given by equation (10), where the upper limit of integration extends to the radius of the inner core.

The energy concept is a useful physical concept when it is described in terms of physical constructs. Geodesic motion of a test particle which is governed by $m(r)$ fits into this category. Proper volume $4\pi r^2 e^{\lambda/2} dr$, rather than coordinate volume $4\pi r^2 dr$, also fits into this category, and this in turn justifies the energy localisation of $T_0{}^0 e^{-\lambda/2}$, which, through integration over proper volume, yields the observable $m(r)$.

The form for ϵ_p, expressed in Schwarzschild coordinates, can be generalised. For a general spherically symmetric metric,

$$ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - R^2 d\Omega^2 \tag{11}$$

where ν, λ, and R are functions of r and t. It can also be shown (Thompson and Whitrow 1967, Cahill and McVittie 1970) that

$$m' = 4\pi R^2 (T_0{}^0 R' - T_1{}^0 \dot{R}) \tag{12}$$

with the mass function, generalised to include cases with time dependence, as

$$m(r, t) = \tfrac{1}{2} R(1 + e^{-\nu} \dot{R}^2 - e^{-\lambda} R'^2) \tag{13}$$

where a prime and a dot denote differentiation with respect to r and t respectively. If we restrict the coordinates to be co-moving, there is no momentum flux and $T_1{}^0$ vanishes. Then, from equation (12), we can write

$$m(r, t) = \int_0^r 4\pi R^2 T_0{}^0 R' \, dr \tag{14}$$

and since the proper volume element is $4\pi R^2 e^{\lambda/2} dr$, the generalised expression for ϵ_p is

$$\epsilon_p = T_0{}^0 R' e^{-\lambda/2}. \tag{15}$$

It is perhaps worthy of remark that for a fluid referred to its rest frame,

$$T_0{}^0 = \epsilon. \tag{16}$$

Since ϵ is a four-scalar, there is perhaps a natural psychological tendency to wish to reject ϵ_p (equation (7)) as the important physical quantity. However, this is a mistake. Although ϵ is a four-scalar, $T_0{}^0$ is not a four-scalar. The two quantities simply happen to be equal when the fluid is referred to its rest frame. The quantity which investigators have really meant to examine, namely $T_0{}^0$ in equations (3), (4), (5) etc, has been replaced by ϵ and granted a higher level of physical significance.

To emphasise the error, consider the situation when the fluid sphere is endowed with a spherically symmetric distribution of charge. Equation (1) still describes the metric and equation (3) is one of the field equations whose solution is given by equation (4) as before. However, there is no longer a vacuum region because of the existence of a radial electric field. The solution for λ and ν exterior to the matter distribution is now the Reissner–Nordström metric (Reissner 1916, Nordström 1918a) with total charge q:

$$e^{\nu} = e^{-\lambda} = 1 - \frac{2m}{r} + \frac{q^2}{r^2} \tag{17}$$

which replaces equation (2). Because of the presence of the electric field, the $T_0{}^0$

component of the energy–momentum tensor is no longer ϵ but rather

$$T_0{}^0 = \epsilon + \frac{E^2}{8\pi} \tag{18}$$

where

$$E(r) = \frac{1}{r^2}\int_0^r \rho \; 4\pi r'^2 \, e^{\lambda/2} \, dr' = \frac{q(r)}{r^2} \tag{19}$$

is the electric field at radius r, ρ is the charge density and $q(r)$ is the amount of charge within the sphere of radius r. Clearly $T_0{}^0$ in this case is no longer a four-scalar. The physical significance of $T_0{}^0$ is in its proper perspective in this example. (Energy considerations for charged fluids will be discussed in greater detail in another paper (Cooperstock and de la Cruz 1978).)

Another way of looking at the problem is through the eigenvalue equation for the energy–momentum tensor,

$$T_{ij}u^j = ku_i \tag{20}$$

where u_i is the four-velocity and k, a scalar, is the eigenvalue. By contracting both sides of equation (20) with u^i, we isolate the eigenvalue

$$k = T_{ij}u^i u^j. \tag{21}$$

For a perfect fluid

$$T_{ij} = (\epsilon + p)u_i u_j - pg_{ij} \tag{22}$$

which, in conjunction with equation (21), yields the eigenvalue $k = \epsilon$. It is also the value of T_{00} when this component is evaluated in the co-moving frame, i.e., $T_{00} = \epsilon$, when $u^i = \delta_0{}^i$.

In the case with charge however, the eigenvalue is

$$k \equiv k_c = \epsilon + \frac{1}{4\pi}(-F_{il}F_k{}^l u^i u^k + \tfrac{1}{4}F_{lm}F^{lm}) \tag{23}$$

where F_{ik} is the Maxwell tensor. As before, it is also the value of T_{00} when this component is evaluated in the co-moving frame, and this now assumes the form as given in equation (18). In general, for an arbitrary frame, the true scalar k_c could be found from the form given in equation (23) but *not* from the form given in equation (19). While it is true that T_{00}, evaluated in the co-moving frame $u^i = \delta_0{}^i$, yields the eigenvalue k which is a scalar, the quantity T_{00} itself is not a scalar.

3. The search for a general form

Having achieved and justified a localisation of energy in the case of spherical symmetry, it is natural to seek a form which is applicable in the absence of spherical symmetry. The work of Nordström (1918b), Tolman (1930) and Møller (1972) appears to offer some hope in this regard. The approach is to express the total energy as an integral over all space of the matter tensor plus energy–momentum pseudotensor:

$$M=\iiint_{-\infty}^{\infty}(T_0{}^0+t_0{}^0)(-g)^{1/2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{24}$$

By the use of Gauss's theorem and the imposition of the condition that the system be quasistatic, this integral is transformed into one which extends only over the matter distribution:

$$m=\iiint(T_0{}^0-T_1{}^1-T_2{}^2-T_3{}^3)(-g)^{1/2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{25}$$

A simpler derivation which circumvents the pseudotensor is provided by Landau and Lifshitz (1975). This proceeds from the form of the $R_0{}^0$ component of the Ricci tensor for a time-independent metric, Gauss's theorem, the asymptotic form of the metric and the Einstein field equations.

Applied to a perfect fluid sphere, equation (25) can be expressed as

$$m=\iiint(\epsilon+3p)(-g)^{1/2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{26}$$

where ϵ is the fluid energy density. This form led Whittaker (1968) to seek and find the solution for what he termed a body with 'constant gravitational mass density',

$$\epsilon+3p=\text{constant}. \tag{27}$$

Although the Whittaker solution was nicely derived and analysed, the identification of $\epsilon+3p$ with 'gravitational mass density', or proper total energy density in our terminology, would be within the realm of justification if $(-g)^{1/2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ were the element of proper three-volume rather than the true value $({}^{(3)}g)^{1/2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. Indeed Møller (1972) has performed the division of equation (25) as

$$m=\int(T_0{}^0-T_1{}^1-T_2{}^2-T_3{}^3)(g_{00})^{1/2}\,\mathrm{d}V_{\text{proper}} \tag{28}$$

and has identified

$$\epsilon_{\mathrm{M}}\equiv(T_0{}^0-T_1{}^1-T_2{}^2-T_3{}^3)(g_{00})^{1/2} \tag{29}$$

(or $(\epsilon+3p)(g_{00})^{1/2}$ in the case of fluids with spherical symmetry) as the 'density of gravitational mass'. Moreover, Møller made the interesting observation that 'since ϵ_{M} obviously behaves like a scalar for all purely spatial transformations this interpretation has a well defined physical meaning'. The two characteristics of ϵ_{M} for static and quasistatic situations, that it is general and not confined to systems with spherical symmetry, and that it is endowed with a manifestly scalar character under spatial transformations (as mentioned earlier, the largest extent of invariance which one could hope for), make it a prime candidate for the generalisation which is being sought. Moreover, it is even suggestive that applied to spherical symmetry, ϵ_{M} would coincide with ϵ_{p}. Indeed, Møller shows that for the interior Schwarzschild solution, generated by the equation of state (equation (9)), this is the case.

However, this is an accidental occurrence, peculiar to the Schwarzschild fluid. The authors have examined the Whittaker metric, some Tolman metrics (Tolman 1939) and their own metric generated by the equation of state (equation (8)). Although the total energy for these bodies, whether one performs the computation with ϵ_p or ϵ_M, is necessarily unique, in no case was there the same localisation of energy as in the Schwarzschild fluid. Thus, one is forced, albeit reluctantly, to reject the ϵ_M form of localisation. There are good physical grounds for accepting the ϵ_p localisation in the case of spherical symmetry and since ϵ_M does not reduce to ϵ_p when particularised to spherical symmetry, it is unacceptable.

It would be very interesting to find the correct generalisation of ϵ_p to the case where one does not have spherical symmetry. If the energy localisation is meaningful in spherically symmetric configurations, it is surely meaningful in non-spherically symmetric configurations which are static or quasistatic, and hence do not present the energy ambiguities inherent in systems with gravitational radiation.

Finally, it should be noted that when one is dealing with radiative fields, it is useful to employ one or another form of gravitational energy–momentum pseudotensor, whose integrated values have well defined meaning. However, one would be hard pressed to choose one pseudotensor over another, except possibly with regard to considerations of angular momentum (Landau and Lifshitz 1975). Similarly, one could attempt to choose some particular pseudotensor to ascribe a gravitational field energy density localisation. In general, this would imply a variation in energy content through vacuum regions. However, this would be in conflict with the energy content which one would wish to ascribe from a consideration of m which is perceived by a test particle. For example in V_I of § 2, the motion a test particle is governed by the same m at every point. From the point of view of physical perception, it would appear most logical and useful to ascribe a localisation within the body as previously discussed.

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